A presentation of cnoidal wave theory for practical application

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Cnoidal wave theory is appropriate to periodic waves progressing in water whose depth is less than about one-tenth the wavelength. The leading results of existing theories are modified and given in a more practical form, and the graphs necessary to their use by engineers are presented. As well as results for the wave celerity and shape, expressions and graphs for the water particle velocity and local acceleration fields are given. A few comparisons between theory and laboratory measurements are included.

1. Introduction

Mathematical arguments show that the theory of surface waves, commonly known as Stokes's waves, is most useful (i.e. valid without unreasonable restriction as to the wave amplitude) when the depth to wavelength ratio d/L is greater than about 1/8 or 1/10 (Keulegan 1950; De 1955). For shallower water the theory for a wave type known as cnoidal appears to be more satisfactory. The formula for the wave profile involves the Jacobian elliptic function cn u; hence the term 'cnoidal', analogous to 'sinusoidal'. Keulegan has pointed out that the validity of this theory rests on the assumption that the square of the inclination of the water surface is small in comparison with unity. (On the other hand, for Stokes's waves to be valid in shallow water, the wave amplitude is required to be excessively small, thus making the theory unrealistic.) The theory of cnoidal waves has not been developed in the literature to the state where it can be used extensively by engineers; and the object of the present paper is to fill this need.

Korteweg & de Vries (1895) initiated the theory of cnoidal waves. They showed that it accounts for a general class of long waves of permanent type and finite amplitude; one limiting case of the theory gives the solitary wave, while another limiting case gives sinusoidal waves as accounted for by linearized wave theory. Keulegan & Patterson (1940) have studied the cnoidal wave on somewhat different lines. Again, Keller (1948) treated the problem using the general non-linear shallow-water theory and obtained formulae which are similar to those of Korteweg & de Vries. Littman (1957) has proved the existence of permanent periodic waves of the type in question. The approximate region of validity of the existence proof of cnoidal waves by Littman is shown in figure 1. Benjamin & Lighthill (1954) have advanced the theory considerably with regard to the formation of bores and hydraulic jumps, and Iwasa (1955) has also considered it.

A comparison between the various alternative developments of the theory will not be attempted here; rather, we select those results which appear to be most useful and proceed from them to derive data for practical application.

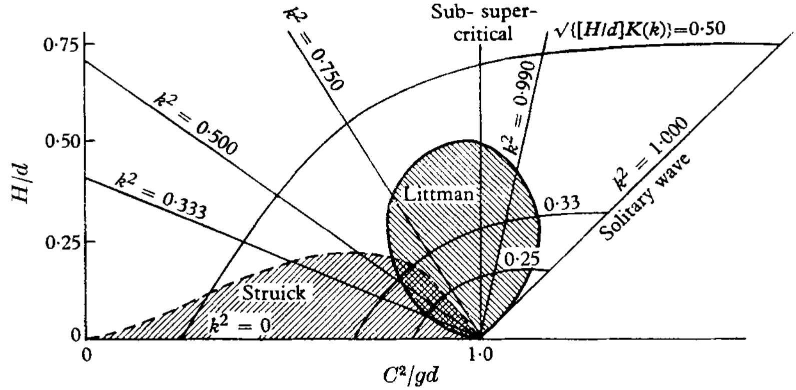

FIGURE 1. Approximate regions of validity of existence proofs by Struik for Stokes's waves and by Littman for cnoidal waves (after Littman 1957).

2. Presentation of the theory

Kortweg & de Vries, Keulegan & Patterson, and Keller use different symbols; however, the critical formulae obtained by them are essentially the same. The wavelength is given by

$$\frac{L}{d} = \frac{4}{\sqrt{3}} K(k) \left(2\bar{L} + 1 - \frac{y_t}{d}\right)^{-\frac{1}{2}}, \tag{1}$$

where d is the still water depth, $K(k)$ is the complete elliptic integral of the first kind with modulus k (it should be noted that $K(k)$ is sometimes denoted by $F_1(k)$). y_t is the vertical distance from the ocean bottom to the wave trough and $\bar{L}$ and k are defined by the following two equations:

$$k^2 = \frac{(y_c/d) - (y_t/d)}{2\bar{L} + 1 - (y_t/d)}, \tag{2}$$

$$\left(2\bar{L} + 1 - \frac{y_t}{d}\right) E(k) = \left(2\bar{L} + 2 - \frac{y_c}{d} - \frac{y_t}{d}\right) K(k), \tag{3}$$

where y_c is the distance from the ocean bottom to the wave crest and $E(k)$ is the complete elliptic integral of the second kind with modulus k. The following inequalities must also hold:

$$2\bar{L} + 1 > \frac{y_c}{d} > \frac{y_t}{d} \quad \text{and} \quad 0 < k^2 \leqslant 1. \tag{4}$$

In using tables of elliptic integrals and functions, the reader is warned that they are often tabulated as functions of the parameter m, where $m = k^2$. Equation (2) can be written

$$\left(2\bar{L} + 1 - \frac{y_t}{d}\right) = \frac{(y_c/d) - (y_t/d)}{k^2} = \frac{H/d}{k^2}. \tag{5}$$

Substituting this into equation (1) and squaring gives

$$\frac{L^2 H}{d^3} = \tfrac{16}{3}[kK(k)]^2. \tag{6}$$

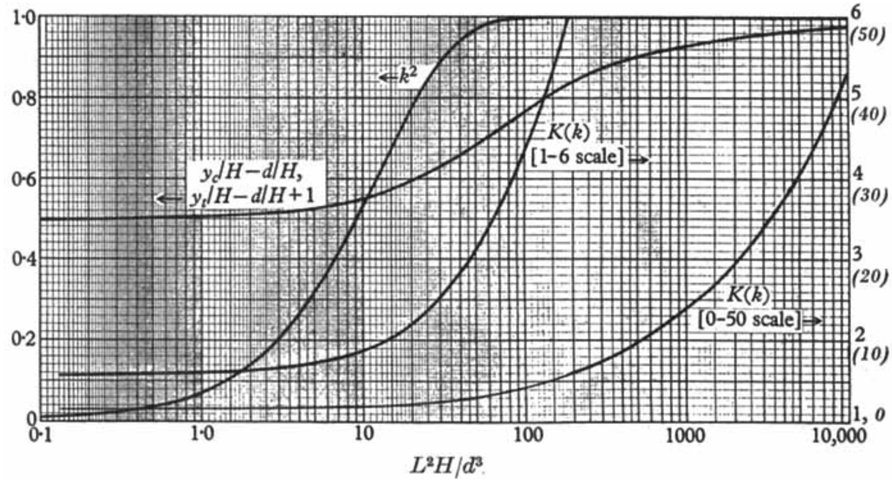

FIGURE 2. Curves obtained from equations (6), (10) and (11*b*), showing relationships between L^2H/d^3 and k^2, $K(k)$, and $y_c/H-d/H = y_t/H-d/H+1$.

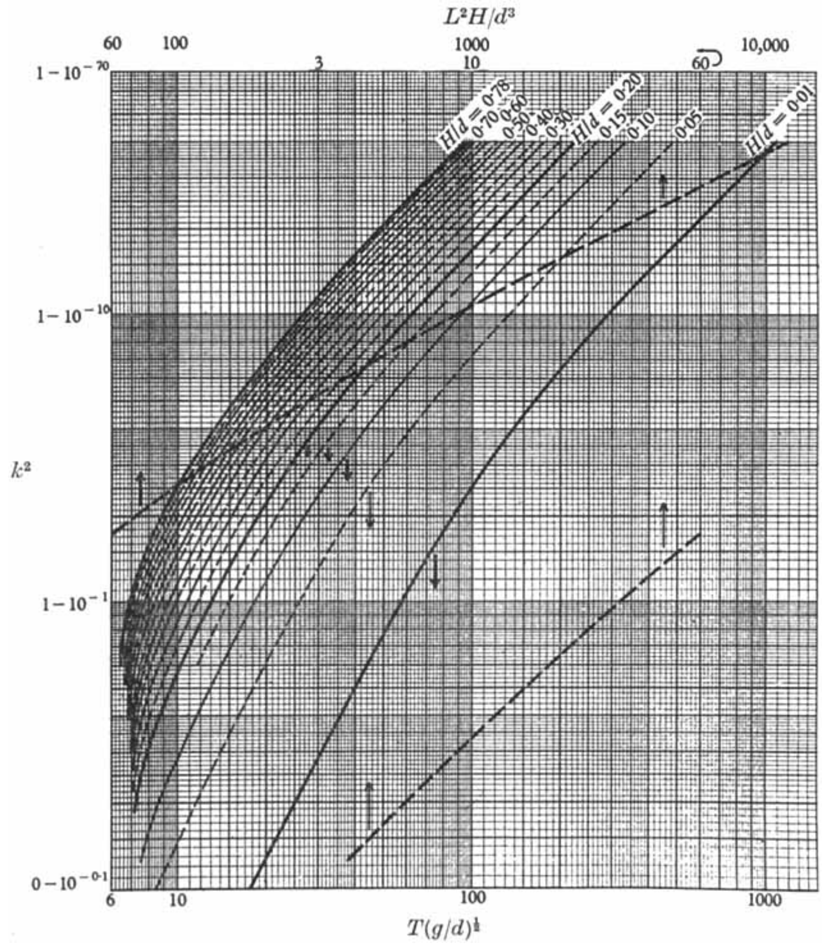

FIGURE 3. Curves showing relationships k^2 *vs* L^2H/d^3, and k^2 *vs* $T(g/d)^{\frac{1}{2}}$ and H/d.

Both Keller (1948) and Littman (1957) also obtain this relationship as their basic approximate solution. L^2H/d^3 is plotted as a function of k^2 in figures 2 and 3. If the wavelength, wave height and water depth are known, then the many formulae of the cnoidal wave theory can be used as they are expressed in terms of various functions of the square of the modulus k. The terminology of the elliptic functions and integrals as used herein are as used by Milne-Thompson (1950).

The wavelength is

$$L = \left(\frac{16d^3}{3H}\right)^{\frac{1}{2}} kK(k). \tag{7}$$

Equation 3 can be rearranged to give

$$E(k) - K(k) = \frac{(1-[y_c/d])}{(2\bar{L}+1-[y_t/d])} K(k). \tag{8}$$

Substituting (1) into the above equation gives

$$\frac{y_c}{d} = \frac{16d^2}{3L^2}\{K(k)\,[K(k)-E(k)]\}+1; \tag{9}$$

or, multiplying by d/H, we get

$$\frac{y_c}{H} - \frac{d}{H} = \frac{16d^3}{3L^2H}\{K(k)\,[K(k)-E(k)]\}. \tag{10}$$

Next, y_t can be obtained from the relationship

$$\frac{y_t}{d} = \frac{y_c}{d} - \frac{H}{d} = \frac{16d^2}{3L^2}\{K(k)\,[K(k)-E(k)]\}+1-\frac{H}{d}, \tag{11a}$$

or

$$\frac{y_t}{H} - \frac{d}{H} + 1 = \frac{16d^3}{3L^2H}\{K(k)\,[K(k)-E(k)]\}. \tag{11b}$$

These equations have been plotted in figures 2 and 3. The relationships among k^2, $K(k)$ and $E(k)$ have been tabulated over the range $k^2 = 1\text{–}10^{-6}$ by Kaplan (1946, 1948) and partially by Hayashi (1930, 1933) and Airey (1935). In order to extend these functions to the range needed for the study of waves ($k^2 = 1\text{–}10^{-40}$) the following equations were used (Jahnke & Emde 1945):

$$K(k) = \Lambda + \tfrac{1}{4}(\Lambda-1)\,k'^2 + \tfrac{9}{64}(\Lambda-\tfrac{7}{6})\,k'^4 + \tfrac{25}{256}(\Lambda-\tfrac{37}{30})\,k'^6 + \ldots, \tag{12a}$$

$$E(k) = 1 + \tfrac{1}{2}(\Lambda-\tfrac{1}{2})\,k'^2 + \tfrac{3}{16}(\Lambda-\tfrac{13}{12})\,k'^4 + \tfrac{15}{128}(\Lambda-\tfrac{6}{5})\,k'^6 + \ldots, \tag{12b}$$

where $k' = \sqrt{(1-k^2)}$ and Λ is the natural logarithm of $4/k'$.

The wave profile is

$$y_s = y_t + H\,\mathrm{cn}^2\left[2K(k)\left(\frac{x}{L}-\frac{t}{T}\right), k\right], \tag{13}$$

where cn is the Jacobian elliptic function associated with cosine. The function cn is singly periodic provided k is real number and $0 \leqslant k < 1$. The period becomes infinite when $k = 1$ (in which case we have the solitary wave). While the period of the cn function is $4K(k)$, the period of the cn^2 function is $2K(k)$. The expression $\mathrm{cn}^2[2K(k)\,(x/L-t/T), k]$ is plotted in figures 4 and 5 as a function of x/L, t/T, with k^2 as parameter. Values of the cn function are available over a limited range of k^2 (Milne-Thompson 1950; Spenceley & Spenceley 1947; Schuler & Gabelein 1955). Values for $0{\cdot}95 < k^2 < 1$ were calculated to three decimal places using the following series (Milne-Thompson 1950):

$$\operatorname{cn}(\bar{u} \mid k^2) = \operatorname{sech}\bar{u} - \tfrac{1}{4}k'^2 \tanh\bar{u}\operatorname{sech}\bar{u}(\sinh\bar{u}\cosh\bar{u} - \bar{u}), \tag{14}$$

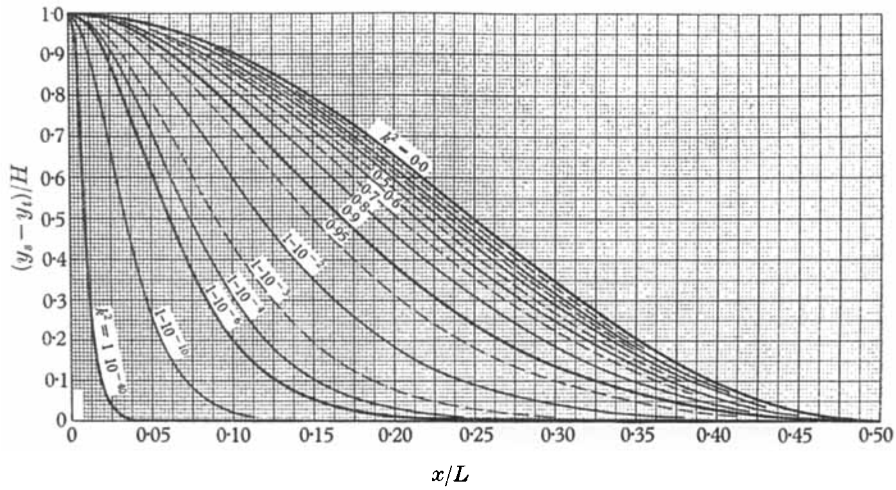

FIGURE 4. Surface profiles of cnoidal waves, obtained from equation (13).

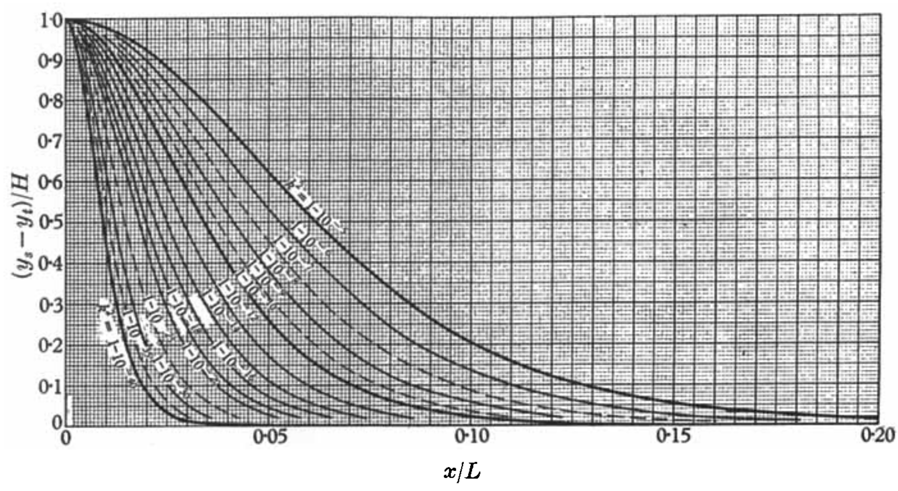

FIGURE 5. Surface profiles of cnoidal waves, obtained from equation (13).

where $\bar{u}$ is the incomplete elliptic integral of the first kind. x/L is the same as $\bar{u}/2K(k)$ for the cn^2 function. $\bar{u}$ has been used rather than the more commonly accepted symbol u to avoid confusion with the horizontal component of water-particle velocity. In figure 6 the theoretical surface profile is compared with some measurements made by Taylor (1955). It can be seen that the cnoidal theory predicts the wave profile very well.

It is interesting to note that when the modulus k is zero,

$$\operatorname{cn}(\bar{u} \mid k) = \operatorname{cn}(\bar{u} \mid 0) = \cos\bar{u} \quad \text{and} \quad K(k) = \tfrac{1}{2}\pi;$$

hence $4K(k) = 2\pi$, and the wave profile is given by the trignometric functions. When $k = 1$, $\mathrm{cn}(\bar{u} \mid 1) = \mathrm{sech}\,\bar{u}$ and we have the hyperbolic function with $K(k) = \infty$; hence the period becomes infinite and we then have the solitary wave. It can be seen that when k is reduced from 1 to 0·9999 the period $4K$ is reduced from infinity to about 7π, whereas the further reduction of k to 0 reduces $4K$ to only 2π.

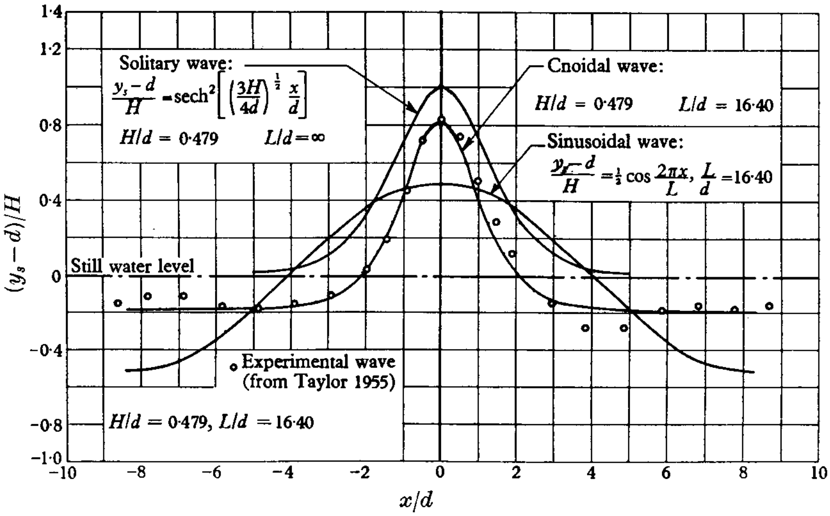

FIGURE 6. Comparison between measured and theoretical wave profiles.

The wave velocity (using Stokes's second definition of wave velocity, which is the velocity of the propagation of the wave-form when the horizontal momentum of the liquid has been reduced to zero by the addition of a uniform motion) is

$$C = (gd)^{\frac{1}{2}}\left[1 + \frac{H}{d}\frac{1}{k^2}\left(\frac{1}{2} - \frac{E(k)}{K(k)}\right)\right], \tag{15a}$$

or, as $\dfrac{16d^3}{3L^2H}K^2(k) = \dfrac{1}{k^2}$,

$$C = (gd)^{\frac{1}{2}}\left[1 + \frac{16d^2}{3L^2}K^2(k)\left(\frac{1}{2} - \frac{E(k)}{K(k)}\right)\right]. \tag{15b}$$

Equation (15a) has been plotted in figure 7.

For one limiting case (the solitary wave), k^2 approaches unity, (k) is unity and $K(k)$ is infinity; hence

$$C = (gd)^{\frac{1}{2}}\left(1 + \frac{H}{2d}\right). \tag{15c}$$

This approximation to

$$C = \left\{gd\left(1 + \frac{H}{d}\right)\right\}^{\frac{1}{2}} \tag{15d}$$

is higher by a maximum of only 2 % even for the case of the solitary wave of maximum steepness ($H/d = 0·78$).

For the other limiting case (the linear theory, where $k^2 \to 0$), $E(k)/K(k) \to$ unity and

$$C = (gd)^{\frac{1}{2}}\left(1 - \frac{2\pi^2d^2}{3L^2}\right), \tag{15e}$$

which is an approximation to

$$C = \left\{gd\left(1-\frac{4\pi^2 d^2}{3L^2}\right)\right\}^{\frac{1}{2}}. \qquad (15f)$$

Now, the linear theory gives (Lamb 1945, p. 366)

$$C = \left\{\frac{gL}{2\pi}\tanh\frac{2\pi d}{L}\right\}^{\frac{1}{2}}. \qquad (15g)$$

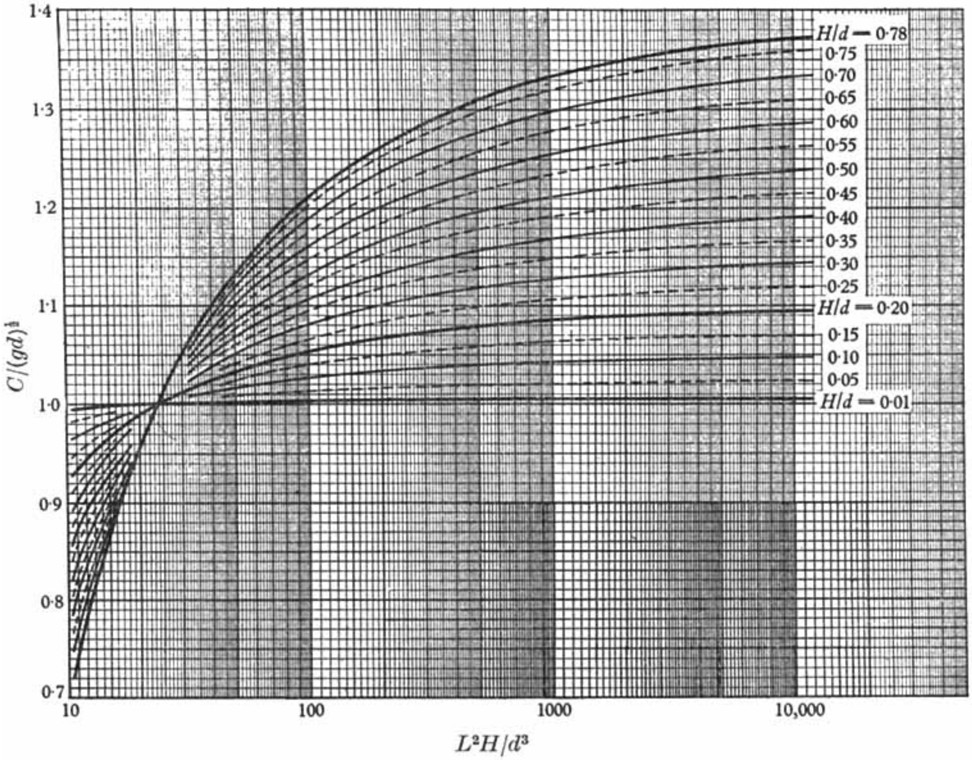

FIGURE 7. Curves obtained from equation (15a), showing relationship between the cnoidal wave velocity (Korteweg & de Vries) and L^2H/d^3.

But the first two terms in the expansion of tanh $(2\pi d/L)$ are

$$\frac{2\pi d}{L}-\frac{8\pi^2 d^3}{3L^3}.$$

So

$$C \doteqdot \left\{gd\left(1-\frac{4\pi^2 d^2}{3L^2}\right)\right\}^{\frac{1}{2}}. \qquad (15h)$$

which is in agreement with equation (15 f).

The equation for wave celerity as given by Keulegan & Patterson (1940), and Littman (1957), which is the velocity of the wave crest with respect to fixed co-ordinates, can be written

$$C^2 = gd\left\{1+\frac{H}{d}\left[-1+\frac{1}{k^2}\left(2-3\frac{E(k)}{K(k)}\right)\right]\right\}. \qquad (16)$$

This equation has been plotted in figure 8 as a function of L^2H/d^3, and in figure 9 as a function of $T(g/d)^{\frac{1}{2}}$.

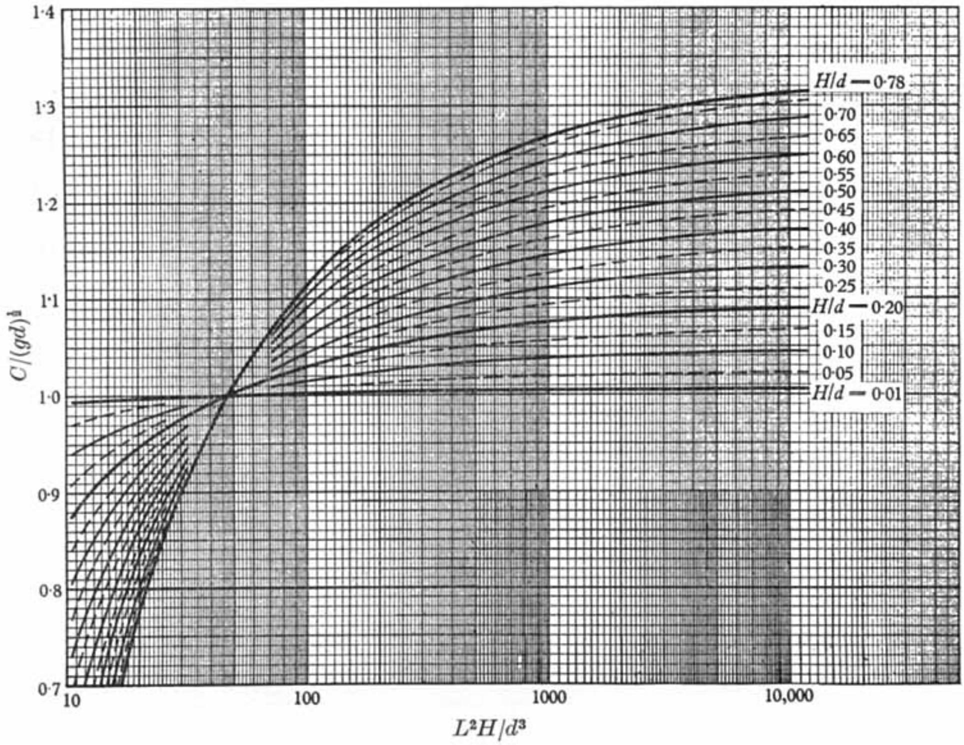

Figure 8. Curves obtained from equation (16), showing relationship between the cnoidal wave velocity (Keulegan & Patterson) and L^2H/d^3.

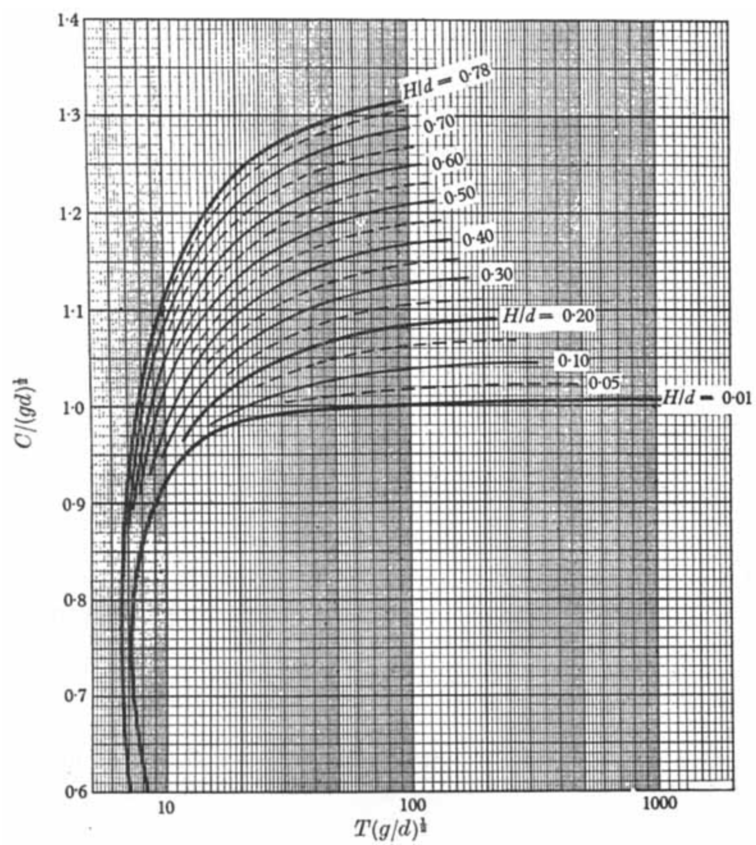

Figure 9. Cnoidal wave velocity (Keulegan & Patterson) as a function of $T(g/d)^{\frac{1}{2}}$.

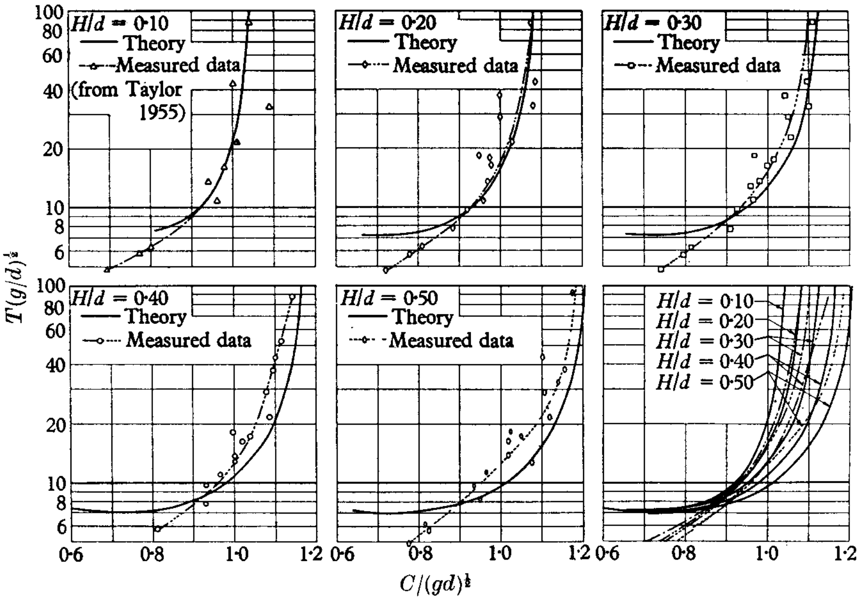

FIGURE 10. Comparisons between theoretical and measured wave velocities.

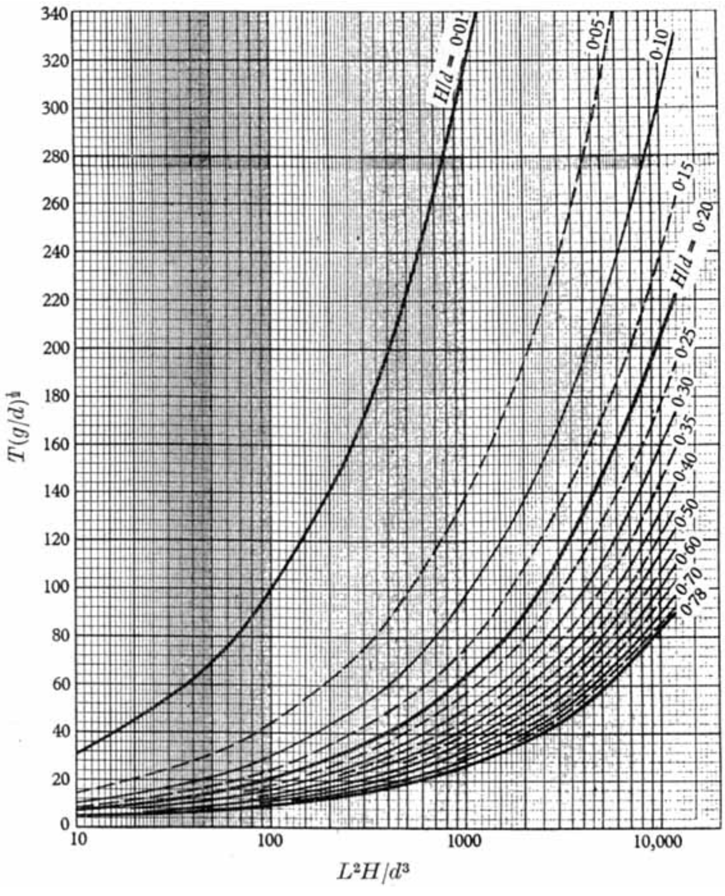

FIGURE 11. Relationships among $T(g/d)^{\frac{1}{2}}$, L^2H/d^3 and H/d (Korteweg & de Vries).

In figure 10 are shown comparisons between the wave velocity as obtained from equation (16) and some measurements of waves in the laboratory.

As cnoidal waves are periodic and of permanent form,

$$L = CT, \quad T = L/C;$$

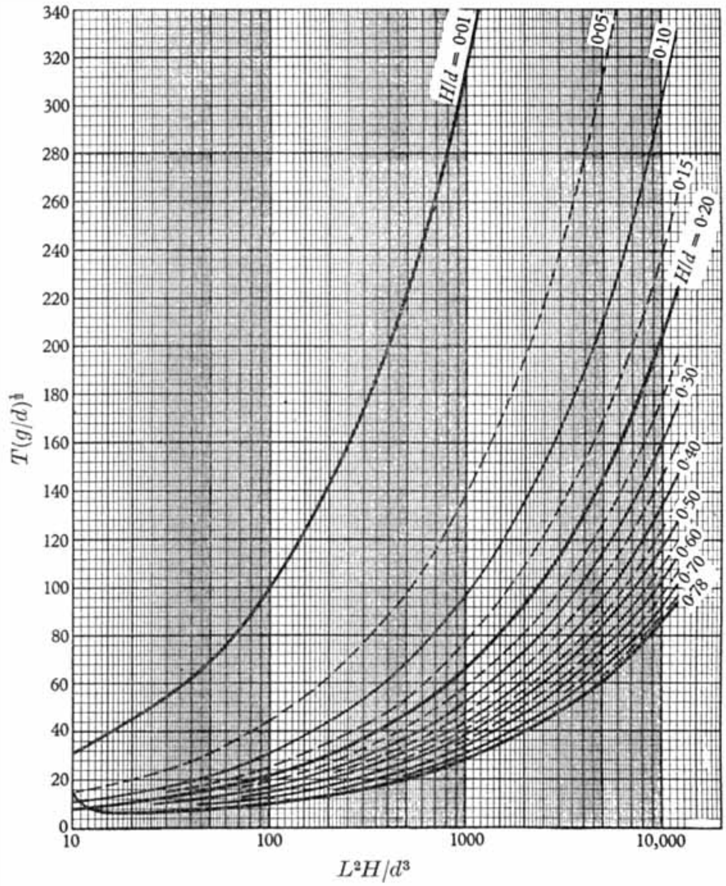

FIGURE 12. Relationships among $T(g/d)^{\frac{1}{2}}$, L^2H/d^3 and H/d (Keulegan & Patterson).

and the wave period is given by

$$T\left(\frac{g}{d}\right)^{\frac{1}{2}} = \left(\frac{16d}{3H}\right)^{\frac{1}{2}} \left[\frac{kK(k)}{1 + \dfrac{H}{dk^2}\left(\dfrac{1}{2} - \dfrac{E(k)}{K(k)}\right)}\right], \tag{17}$$

using the velocity as given in equation (15*a*).

From this equation k^2 can be determined as a function of $T(g/d)^{\frac{1}{2}}$ and H/d, and from this L^2H/d^3 can be determined. This has been done and plotted in figure 11.

Using the velocity as given in equation (16) results in

$$T\left(\frac{g}{d}\right)^{\frac{1}{2}} = \left(\frac{16d}{3H}\right)^{\frac{1}{2}} \frac{kK(k)}{\left\{1 + \dfrac{H}{d}\left[-1 + \dfrac{1}{k}\left(2 - \dfrac{3E(k)}{K(k)}\right)\right]\right\}^{\frac{1}{2}}}, \tag{18}$$

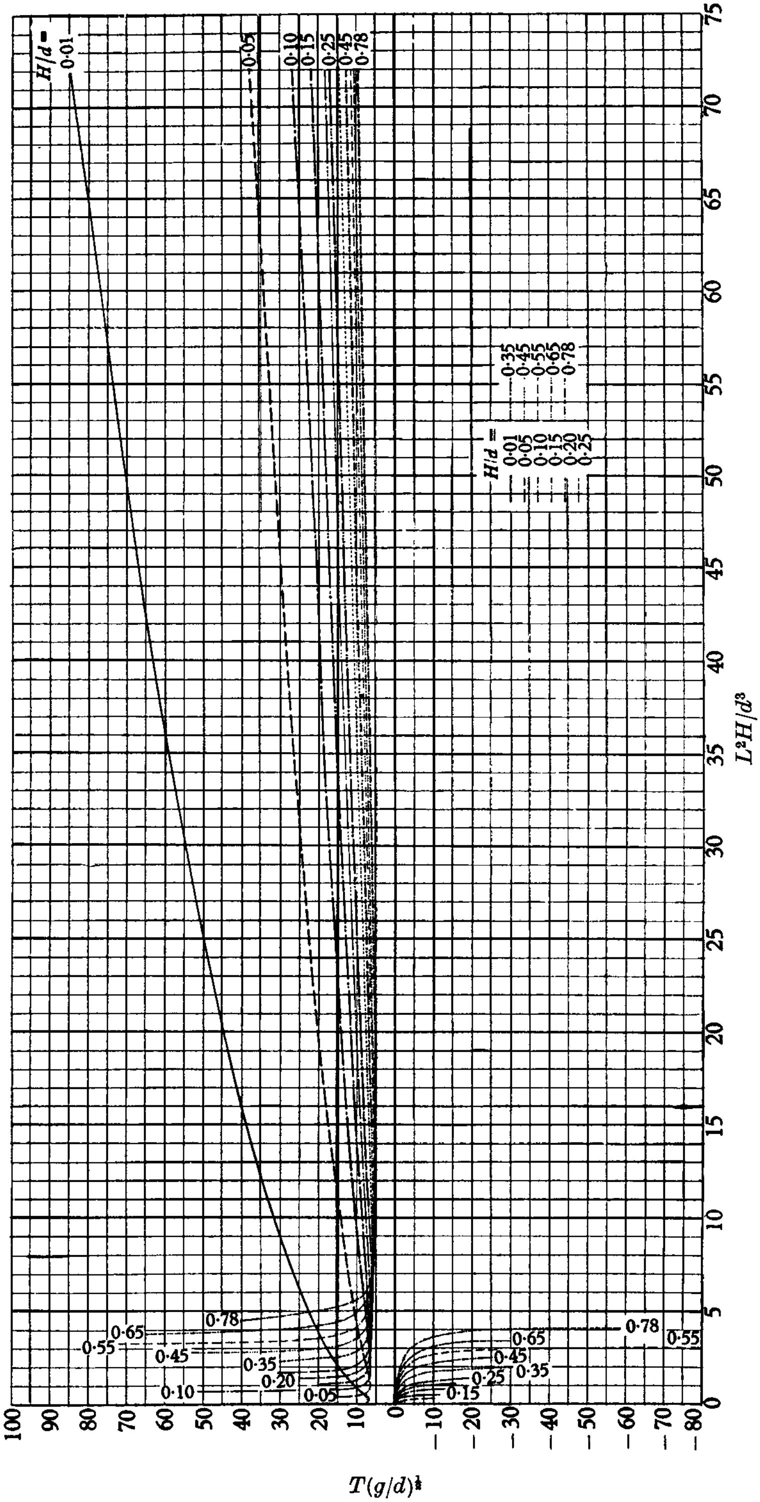

FIGURE 13. Relationships among wave height, wave period, wavelength and water depth for cnoidal waves.

which has been plotted in figure 3 as a function of k^2 and H/d, and in figures 12 and 13 as a function of L^2H/d^3 and H/d. The results in figure 13 show that a wave of a given period and height, and for a given water depth, can have two possible lengths. The physical significance of this is not apparent.

The pressure at any distance y above the bottom has been shown by Keller (1948) to be, to the second approximation,

$$p = \rho g(y_s - y), \tag{19}$$

where ρ is density and y_s is given by (13). It is rather surprising that this simple hydrostatic expression applies.

The horizontal and vertical components of water-particle velocities at any point x, y within the fluid can be obtained from the following equations given by Keulegan & Patterson (1940):

$$\frac{u}{(gd)^{\frac{1}{2}}} = \left[\frac{h}{d} - \frac{h^2}{4d^2} + \left(\frac{d}{3} - \frac{y^2}{2d}\right)\frac{\partial^2 h}{\partial x^2}\right],$$

$$\frac{v}{(gd)^{\frac{1}{2}}} = -y\left[\left(\frac{1}{d} - \frac{h}{2d^2}\right)\frac{\partial h}{\partial x} + \frac{1}{3}\left(d - \frac{y^2}{2d}\right)\frac{\partial^3 h}{\partial x^3}\right],$$

$$h = y_s - d = -d + y_t + H\,\mathrm{cn}^2\left[2K(k)\left(\frac{x}{L} - \frac{t}{T}\right), k\right].$$

These equations become

$$\frac{u}{(gd)^{\frac{1}{2}}} = \left[-\frac{5}{4} + \frac{3y_t}{2d} - \frac{y_t^2}{4d^2} + \left(\frac{3H}{2d} - \frac{y_t H}{2d^2}\right)\mathrm{cn}^2(\;) - \frac{H^2}{4d^2}\mathrm{cn}^4(\;)\right.$$
$$\left. - \frac{8HK^2(k)}{L^2}\left(\frac{d}{3} - \frac{y^2}{2d}\right)(-k^2\,\mathrm{sn}^2(\;)\,\mathrm{cn}^2(\;) + \mathrm{cn}^2(\;)\,\mathrm{dn}^2(\;) - \mathrm{sn}^2(\;)\,\mathrm{dn}^2(\;))\right], \tag{20}$$

$$\frac{v}{(gd)^{\frac{1}{2}}} = y\cdot\frac{2HK(k)}{Ld}\left[1 + \frac{y_t}{d} + \frac{H}{d}\mathrm{cn}^2(\;) + \frac{32K^2(k)}{3L^2}\left(d^2 - \frac{y^2}{2}\right)(k^2\,\mathrm{sn}^2(\;)\right.$$
$$\left. - k^2\,\mathrm{cn}^2(\;) - \mathrm{dn}^2(\;))\right]\mathrm{sn}(\;)\,\mathrm{cn}(\;)\,\mathrm{dn}(\;), \tag{21}$$

where sn ( ) refers to $\mathrm{sn}\,[2K(k)\,(x/L - t/T), k]$, etc. The local accelerations are

$$\frac{\partial u}{\partial t} = (gd)^{\frac{1}{2}}\frac{4HK(k)}{Td}\left[\left(\frac{3}{2} - \frac{y_t}{2d}\right) - \frac{H}{2d}\mathrm{cn}^2(\;) + \frac{16K^2(k)}{L^2}\left(\frac{d^2}{3} - y^2\right)\right.$$
$$\left. \times(k^2\,\mathrm{sn}^2(\;) - k^2\,\mathrm{cn}^2(\;) - \mathrm{dn}^2(\;))\right]\mathrm{sn}(\;)\,\mathrm{cn}(\;)\,\mathrm{dn}(\;), \tag{22}$$

$$\frac{\partial v}{\partial t} = y(gd)^{\frac{1}{2}}\frac{4HK^2(k)}{LTd}\left\{\left[1 + \frac{y_t}{d}\right][\mathrm{sn}^2(\;)\,\mathrm{dn}^2(\;) - \mathrm{cn}^2(\;)\,\mathrm{dn}^2(\;) + k^2\,\mathrm{sn}^2(\;)\,\mathrm{cn}^2(\;)]\right.$$
$$+ \frac{H}{d}[3\,\mathrm{sn}^2(\;)\,\mathrm{dn}^2(\;) - \mathrm{cn}^2(\;)\,\mathrm{dn}^2(\;) + k^2\,\mathrm{sn}^2(\;)]\,\mathrm{cn}^2(\;) - \frac{32K^2(k)}{3L^2}\left[d^2 - \frac{y^2}{2}\right]$$
$$\times[9k^2\,\mathrm{sn}^2(\;)\,\mathrm{cn}^2(\;)\,\mathrm{dn}^2(\;) - k^2\,\mathrm{sn}^4(\;)\,(k^2\,\mathrm{cn}^2(\;) + \mathrm{dn}^2(\;))$$
$$\left. + k^2\,\mathrm{cn}^4(\;)\,(k^2\,\mathrm{sn}^2(\;) + \mathrm{dn}^2(\;)) + \mathrm{dn}^4\,(\mathrm{sn}^2(\;) - \mathrm{cn}^2(\;))]\right\}. \tag{23}$$

In order to use the equations for water-particle velocities the necessary numbers can be obtained from figures 2, 3, 4 and 5, as

$$\bar{u} \mid k \equiv 2K(k)\,x/L, \tag{24}$$

$$\mathrm{sn}^2(\bar{u} \mid k) \equiv 1 - \mathrm{cn}^2(\bar{u} \mid k), \tag{25}$$

$$\mathrm{dn}^2(\bar{u} \mid k) \equiv 1 - k^2[1 - \mathrm{cn}^2(\bar{u} \mid k)]. \tag{26}$$

A few comparisons of theory with laboratory measurements are shown in figure 14. In considering the vertical velocity it should be noted that the curve

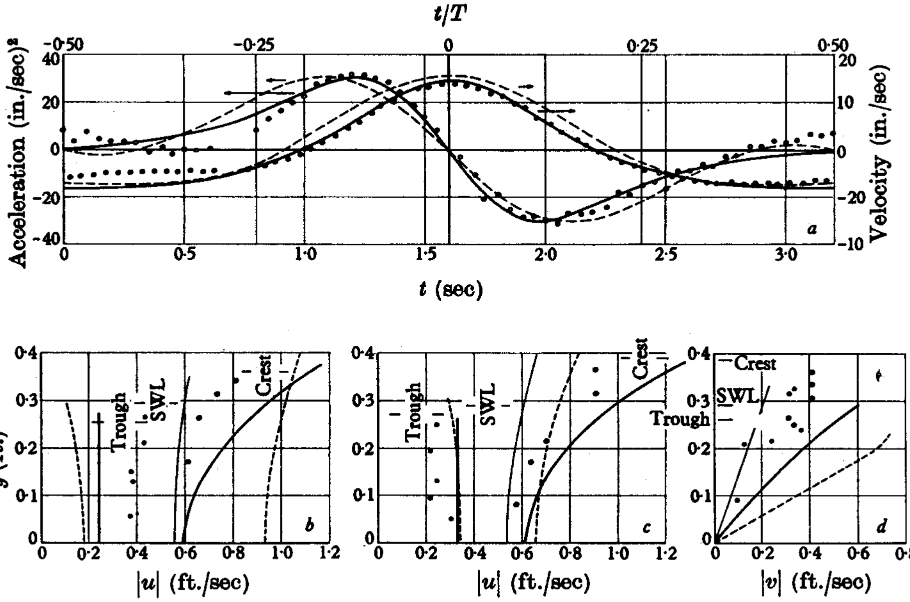

FIGURE 14. Comparisons of horizontal components of water-particle velocity and acceleration with cnoidal wave theory. *a*: ○, Experimental points (Elliott 1953); - - - - -, Stokes's second-order theory; ——, Cnoidal theory. *b* to *d*: •, ○, Experimental points (Morison & Crooke 1953); ——, Linear theory; - - - - -, Stokes's second-order theory; ——, Cnoidal theory. In *b* and *c*, points •, ○ give $|u|$ under wave troughs and crests respectively; in *d*, points •, ○ give $|v|$ under SWL leading wave crests and following wave crests respectively.

	H (ft.)	T (sec.)	d (ft.)	L (ft.)
a	0·483	3·20	2·0	24·7
b	0·105	1·62	0·292	5·10
c	0·120	1·27	0·292	3·71
d	0·137	2·09	0·292	6·58

plotted for the cnoidal theory is for the phase of maximum vertical velocity and this occurs prior to the time the wave profile goes through the still water level.

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REFERENCES

AIREY, J. R. 1935 Toroidal functions and the complete elliptic integrals. *Phil. Mag.* 7th Series, **19**, 177–88.

BENJAMIN, T. B. & LIGHTHILL, M. J. 1954 On cnoidal waves and bores. *Proc. Roy. Soc.* **A**, **224**, 448–60.

DE, S. C. 1955 Contributions to the theory of Stokes's waves. *Proc. Camb. Phil. Soc.* **51**, 713–36.

ELLIOTT, J. G. 1953 Interim report, Contract NOy–12561, Hydrodynamics Laboratory, Calif. Inst. of Tech.

HAYASHI, K. 1930 *Tafeln der besselschen, theta-, Kugel, und anderer funktionen.* Berlin: Springer.

HAYASHI, K. 1933 *Tafeln für die differenzenrechnung sowie für die hyperbel-besselschen, elliptischen, und anderer funktionen.* Berlin: Springer.

IWASA, Y. 1955 Analytical considerations on cnoidal and solitary waves. *Memoirs of Faculty of Engineering, Kyoto Univ.* (*Japan*), **17**, no. 4.

JAHNKE, E. & EMDE, F. 1945 *Tables of Functions with Formulae and Curves*, 4th ed. New York: Dover.

KAPLAN, E. L. 1946 Auxiliary table of complete elliptic integrals. *J. Math. Phys.* **25**, 26–36.

KAPLAN, E. L. 1948 Auxiliary table for the incomplete elliptic integrals. *J. Math. Phys.* **27**, 11–36.

KELLER, J. B. 1948 The solitary wave and periodic waves in shallow water. *Commun. appl. Math.* **1**, 323–39.

KEULEGAN, G. H. 1950 Wave motion. *Engineering Hydraulics. Proc. Fourth Hydraulics Conf.*, ch. 11, pp. 711–68. Ed. by Hunter Rouse. New York: Wiley.

KEULEGAN, G. H. & PATTERSON, G. W. 1940 Mathematical theory of irrotational translation waves. *J. Res. Nat. Bur. Stand., U.S. Dep. Commerce*, **24**, 47–101.

KORTEWEG, D. J. & DE VRIES, G. 1895 On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.* 5th Series, **39**, 422–43.

LAMB, H. 1945 *Hydrodynamics*, 6th ed. New York: Dover.

LITTMAN, W. 1957 On the existence of periodic waves near critical speed. *Commun. Pure appl. Math.* **10**, 241–69.

MILNE-THOMPSON, L. M. 1950 *Jacobian Elliptic Function Tables.* New York: Dover.

MORISON, J. R. & CROOKE, R. C. 1953 The mechanics of deep water, shallow water, and breaking waves. *Tech. Mem. U.S. Army, Corps of Engineers, Beach Erosion Board*, no. 40.

SCHULER, M. & GABELEIN, H. 1955 *Eight and Nine Place Tables of Elliptical Functions.* Berlin: Springer.

SPENCELEY, G. W. & SPENCELEY, R. M. 1947 *Smithsonian Elliptic Functions Tables.* Washington, D.C.: Smithsonian Institution.

TAYLOR, D. C. 1955 An experimental study of the transition between oscillatory and solitary waves, M.S. Thesis, Mass. Inst. of Tech.